

PRPgenerator User's Manual

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1 Introduction

PRP stands for Parallel Road Profiles and therefore PRPgenerator generates parallel synthetic road profiles of a given length. The main objective of the generation of parallel profiles along a road stretch is to take into consideration that road profile is not constant along road width and therefore imperfections under left and right tyres of a 4 (or more) wheeled vehicle are different.

This fact is frequently neglected in road vehicle and bridge dynamics. This simplification can be found both when road profile is measured in situ and also when synthetic profiles are employed. This entails that the road profile is the same under all wheels. Nevertheless, when a four wheeled vehicle runs over a road, left and right wheels do not follow the same path, thus the profiles under each side tyres are different, but those profiles are not independent.

Road roughness is a main source of dynamic excitation in road vehicle-bridge dynamics and has an influence on both systems, the vehicle and the bridge. A correct

and complete definition of the road surface is then a key point in vehicle–bridge dynamic analysis.

Neglecting L-R difference has an effect on both the vehicle and the bridge. With respect to Vehicle dynamics: vertical acceleration is overestimated, body roll is not gathered, forces under wheels are not accurately determined... In regard to Bridge, dynamic effects are overestimated: accelerations, deflection, Dynamic Amplification Factors.... Influence in vehicle dynamic behaviour is addressed in [1], effects on bridges in [2].

PRPgenerator is implemented in MATLAB and it is a stand-alone application so no MATLAB license or installation is required. The software is introduced in [3] even though its theoretical background is explained in [1, 2], this background is contained in this manual as well.

2 Operating system and hardware requirements

Current version of PRPgenerator works in Windows 7/XP/Vista/8. Minimum requirements are expected to be:

- 4 GB RAM
- 64-bit processor

3 Installation of PRPgenerator

1. Download PRPgenerator.rar to your computer.
2. Extract its content at any folder. A new folder called PRPgenerator should appear. It contains one folder called License and two files: PRPgenerator_Manual.pdf and PRPgenerator_pkg.exe.
3. Double click PRPgenerator_pkg.exe and follow the instructions on the screen.
4. Once the installation has finished new files appear in the folder PRPgenerator (see figure 1)
5. Double click PRPgenerator.exe (not PRPgenerator_pkg.exe) to run the software.

Nombre	Fecha de modifica...	Tipo	Tamaño
License	01/02/2013 14:24	Carpeta de archivos	
MCRInstaller.exe	14/08/2011 4:58	Aplicación	423.277 KB
PRPgenerator.exe	01/02/2013 14:16	Aplicación	2.054 KB
PRPgenerator_pkg.exe	01/02/2013 14:18	Aplicación	424.957 KB
readme.txt	01/02/2013 14:16	Archivo TXT	1 KB

Figure 1: PRPgenerator folder with installation completed.

4 Theoretical background

In site-specific problems the actual road profile can be measured and employed in the calculations. When the actual profile of a particular road stretch is not appropriate but a set of profiles that are representative of a certain sort of roads, stochastic definitions for the generation of synthetic profiles are used. This section is divided in two parts. Firstly, a brief explanation of the generation of simple profiles is presented; this is important because it helps to introduce some concepts and assumptions that will be used in the generation of parallel profiles. In second place, the generation of parallel road profiles is explained.

4.1 The classic way: one only profile for every wheel

In this subsection we are interested in how to create rough profiles along a road, these profiles will be independent of each other. Profiles generated in this way are prescribed at every vehicle wheel so the only difference in the imposed vertical displacement at the tyres arises from the time lag between front and rear tyres. This is the common procedure employed in road bridge dynamics so the reader is probably familiar with it.

It is a commonly assumed hypothesis that the randomness of the road surface roughness can be represented with a normal stationary ergodic random process described by means of its Power Spectral Density (PSD), also known as G ([4, 5, 6...]). Several PSD functions have been proposed by different authors. A survey of different approximations can be found in [7].

Once a PSD definition has been chosen, road profiles are generated as the sum of a series of harmonics:

$$r(x) = \sum_i^N \sqrt{2G(n_i)\Delta n} \cos(2\pi n_i x + \phi_i) \quad (1)$$

where r is the road elevation, $G(n)$ is the one-sided power spectral density for the spatial frequency or wavenumber n and ϕ_i is the random phase angle uniformly distributed from 0 to 2π . This expression employs N frequencies between two values n_{min} and n_{max} and the increment value (Δn) is set constant:

$$\Delta n = \frac{n_{max} - n_{min}}{N} \quad (2)$$

ISO-8608 specifications [8] propose a distribution for the one-sided PSD defined by the following expression:

$$G(n) = G(n_0) \left(\frac{n}{n_0} \right)^{-2} \quad (3)$$

where $G(n)$ is the one-sided power spectral density for the spatial frequency or wavenumber n and $G(n_0)$ is the one-sided power spectral density for the reference spatial frequency $n_0 = 0.1 \text{ m}^{-1}$. The value for $G(n_0)$ is prescribed by ISO-8608 [8] as a function of the road class (Table 1).

Road class	$G(n_0)$ [m ³]
A (Very good)	$16 \cdot 10^{-6}$
B (Good)	$64 \cdot 10^{-6}$
C (Medium)	$256 \cdot 10^{-6}$
D (Poor)	$1024 \cdot 10^{-6}$
E (Very poor)	$4096 \cdot 10^{-6}$

Table 1: $G_d(n_0)$ for different road classes according to ISO-8608.

This expression is adopted in **PRPgenerator**. Lower and upper spatial frequency limits are set as $n_{min} = 0.01$ m⁻¹ and $n_{max} = 10.0$ m⁻¹ [8], those limits correspond to wavelength limits $\lambda_{max} = 100$ m and $\lambda_{min} = 0.1$ m.

4.2 The improved way: parallel profiles

When a four wheeled vehicle runs along a road an imposed history of displacement is applied on all its tyres due to the road imperfections. For wheels in the same transverse position, for example the two right wheels, it can be assumed that both run on the same profile. Hence, the imposed displacement is the same but there is a time lag because front tyres reach every point a while before rear ones. On the other hand, tyres that are not transversely aligned, that is to say left and right tyres, run over different paths and therefore the imperfections and the imposed displacements are different.

As stated before, different longitudinal profiles computed by means of equation (1) are independent profiles which correspond to one line along the road.

PRPgenerator assumes two main hypotheses in regard to road surface:

- **Homogeneity:** every straight profile on the road has the same statistical characteristics, independently of its **position**.
- **Isotropy:** every straight profile on the road has the same statistical characteristics, independently of its **direction**.

Therefore, in a homogeneous and isotropic road surface every straight profile has the same statistical characteristics, independently of its direction or position. This entails that the surface could be completely described with the PSD of any straight profile.

Consider any surface $r(x, y)$ where r is the distance to the Oxy plane and two parallel profiles $r_L(x)$ y $r_R(x)$ (Left and Right) that are the intersections of the surface with the vertical planes $y = b$ and $y = -b$ respectively (figure 2). This two profiles are defined as functions of x , which is the road longitudinal coordinate. The autocorrelation functions of these profiles $R_L(\delta)$ and $R_R(\delta)$ in this kind of process fulfill the following condition:

$$R_L(\delta) = R_R(\delta) = R(\delta) \quad (4)$$

Cross-correlation functions for these profiles will be also equal:

$$R_{LR}(\delta) = R_{RL}(\delta) = R_x(\delta) \quad (5)$$

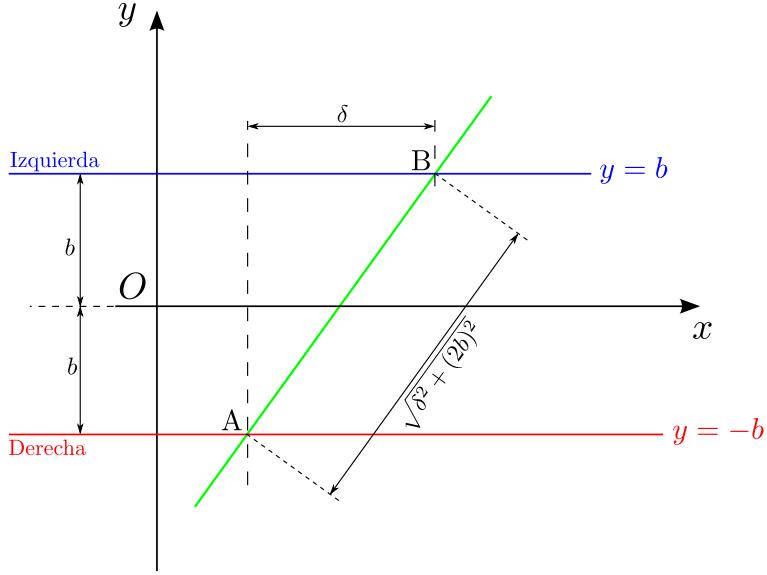


Figure 2: Parallel profiles in a homogeneous and isotropic surface.

The cross-correlation of the two profiles $R_x(\delta)$ is obtained with the elevation of two points, one in each profile, at a distance from each other equal to δ in direction x ; that is to say $r_L(x_B)$ and $r_R(x_A)$. Those points are also related in the autocorrelation function of the straight profile taken along line AB, the autocorrelation of AB profile is the same as that of the Left and Right profiles r_L and r_R since it is also a straight profile on the same homogeneous and isotropic surface. Therefore:

$$R_x(\delta) = R(\sqrt{\delta^2 + (2b)^2}) \quad (6)$$

where $\sqrt{\delta^2 + (2b)^2}$ is the distance along AB profile between two points at δ in x direction, b is half the distance between the two profiles.

PSD function $G(n)$ that describes the profiles must satisfy some conditions in order to be suitable for describing an isotropic surface, it must be monotonically non-increasing and have a bounded integral [9]. Expression proposed in ISO-8608 (equation (3)) does not satisfy isotropy conditions and therefore it has to be slightly modified. If the function is assumed constant at all spatial frequencies below n_a we will get a piecewise-defined spectral density function that is valid for the description of an isotropic surface:

$$G(n) = \begin{cases} G(n_0) \left(\frac{n_a}{n_0}\right)^{-2} & \text{if } n \leq n_a \\ G(n_0) \left(\frac{n}{n_0}\right)^{-2} & \text{if } n > n_a \end{cases} \quad (7)$$

where n_a is set $n_a = 0.01 \text{ m}^{-1}$.

Hence, PSD function gets the appearance shown in figure 3.

Autocorrelation function can be derived from PSD function as:

$$R(\delta) = \int_0^\infty G(n) \cos(2\pi\delta n) dn \quad (8)$$

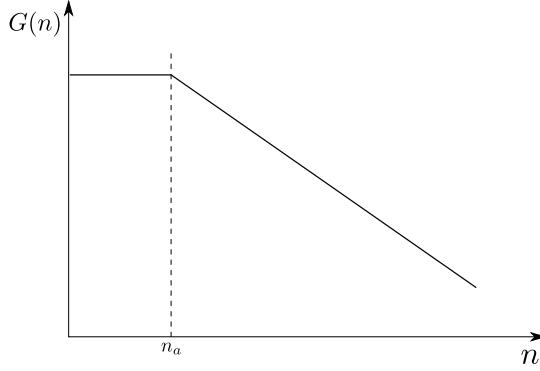


Figure 3: ISO-8608 spectral density function modified in order to represent a homogeneous and isotropic process (logarithmic scale in both axes).

And cross-PSD $G_x(n)$ can be obtained from $R_x(\delta)$ by means of Fourier transform.

Let's see how to get autocorrelation and cross correlation functions from $G(n)$. Given that $G(n_0)$ and n_0 are constant expression 7 can be rewritten as:

$$G(n) = \begin{cases} cn_a^{-2} & \text{if } n \leq n_a \\ cn^{-2} & \text{if } n > n_a \end{cases} \quad (9)$$

with $c = \frac{G(n_0)}{n_0^{-2}}$. This simplification makes the explanation easier.

Substituting equation 9 into equation 8 the following expression is obtained:

$$R(\delta) = 2c \left(n_a^{-1} \frac{\sin(2\pi n_a \delta)}{(2\pi n_a \delta)} + \int_{n_a}^{\infty} n^{-2} \cos(2\pi n \delta) dn \right) \quad (10)$$

And considering expressions (6) and 10:

$$R_x(\delta) = 2c \left(n_a^{-1} \frac{\sin(2\pi n_a \sqrt{\delta^2 + (2b)^2})}{(2\pi n_a \sqrt{\delta^2 + (2b)^2})} + \int_{n_a}^{\infty} n^{-2} \cos(2\pi n \sqrt{\delta^2 + (2b)^2}) dn \right) \quad (11)$$

Integrals in expressions 10 and 11 are solved semi-analitically as explained in appendix A. This solution is computationally faster than the numerical integration.

Once we have computed the cross-correlation function, cross PSD is computed by means of Discrete Fourier Transform:

$$G_x(n_k) = 2 \sum_{n=0}^{N-1} R_x(\delta_n) e^{-\frac{2\pi i}{N} kn} \quad k = 0, 1, \dots, N-1 \quad (12)$$

Since R_x is unique for a pair of profiles, G_x is also unique.

Coherence of a pair of profiles can be expressed in this kind of process as:

$$g(n) = \frac{|G_x(n)|}{G(n)} \quad (13)$$

given that $G_x(n)$ is real [9].

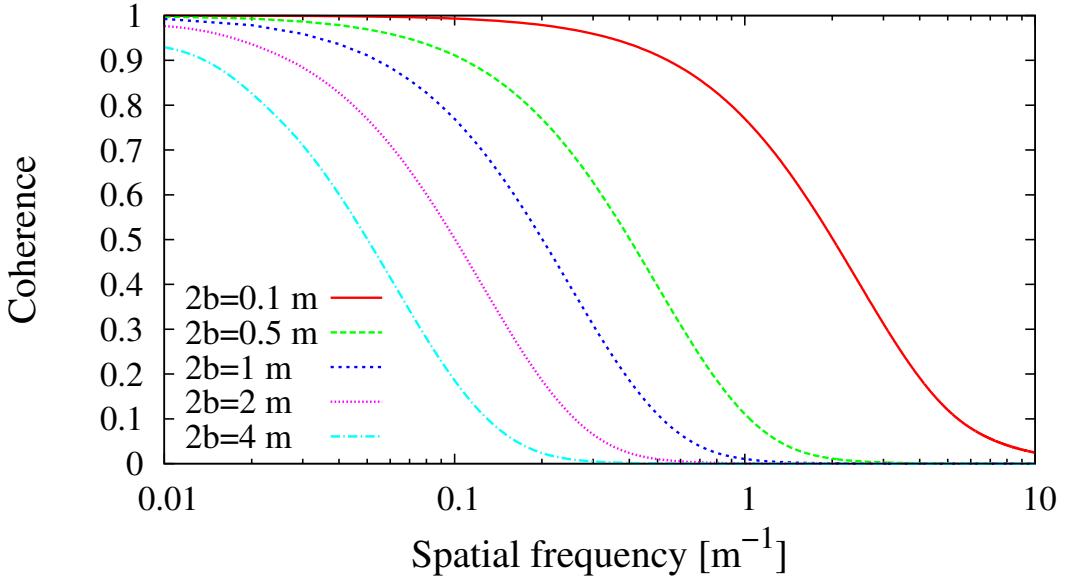


Figure 4: Coherence function for different distances between profiles.

Figure 4 shows coherence functions for different distances between profiles. Actually, coherence function defined in expression (13) is independent of c , therefore it will not depend on the road class but only on the distance between profiles. Coherence is higher for small spatial frequencies or high wavelengths and gets significantly lower for high spatial frequencies or reduced wavelengths. This is quite feasible since high wavelength imperfections affect to a higher width than small wavelength imperfections that are more localized. We can also appreciate that coherence is lower for all wavelengths if the distance between profiles is increased which is also logical.

Once cross-PSD $G_x(n)$ has been computed the two parallel profiles are obtained as:

$$r_L(x) = \sum_i^N \sqrt{2G(n_i)\Delta n} \cos(2\pi n_i x + \phi_i) \quad (14a)$$

$$r_R(x) = \sum_i^N \left[\sqrt{2G(n_i)\Delta n} \cos(2\pi n_i x + \phi_i) + \sqrt{2(G(n_i) - G_x(n_i))\Delta n} \cos(2\pi n_i x + \theta_i) \right] \quad (14b)$$

where ϕ_i and θ_i are random phase angles uniformly distributed from 0 to 2π .

Figure 5 shows parallel Class A profiles at different distances generated with PRPgenerator. The PSD of those profiles is computed, smoothed according to ISO-8608 and compared with the ISO definition (Eq. (3)). The lower is the distance between profiles the higher is the similarity, as it is expected. As can be seen PSD agreement is excellent. If the road stretch length is higher the computation of PSD will be more accurate as can be seen in figure 6 where the road section considered is 400 meters long.

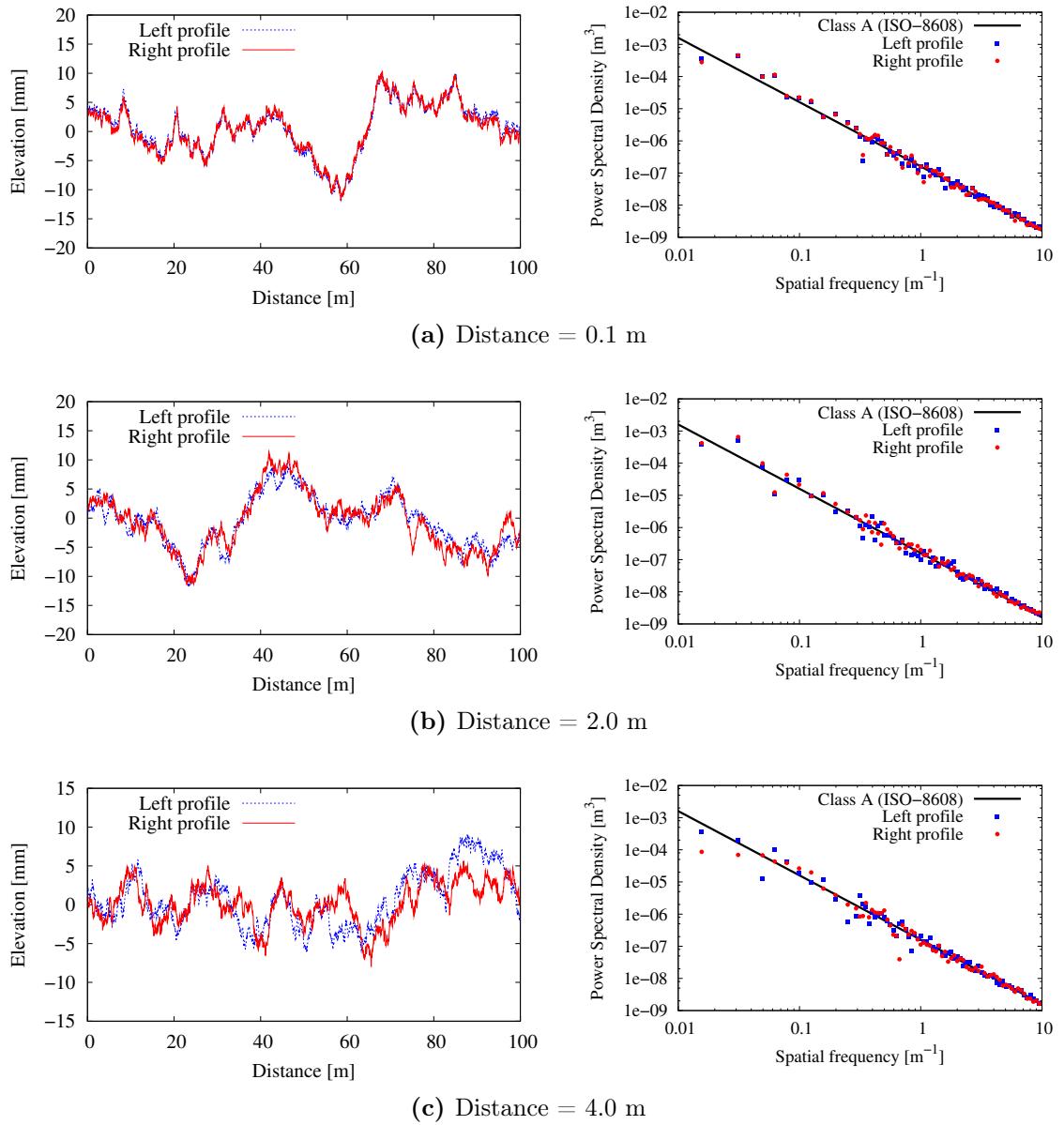


Figure 5: Parallel Class A Road Profiles and their Power Spectral Densities.

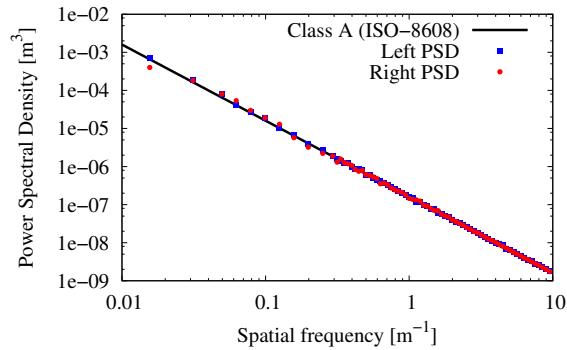


Figure 6: Power Spectral Density of two 400 meters long parallel road profiles (class A).

5 Overview

PRPgenerator is quite simple and intuitive. Its main and only window is shown in figure 7. We can distinguish the following parts:

- PRPgenerator logo on the left-hand upper corner.
- *Problem data* section on the left side. Here is where the user defines the properties of the profiles that are to be generated.
- *Results* section on the right-hand side. In this section some results are depicted: coherence function in the upper window and the generated profiles in the lower one.
- Two buttons, **COMPUTE g** and **GENERATE PROFILES**, are located on the left-hand lower corner.

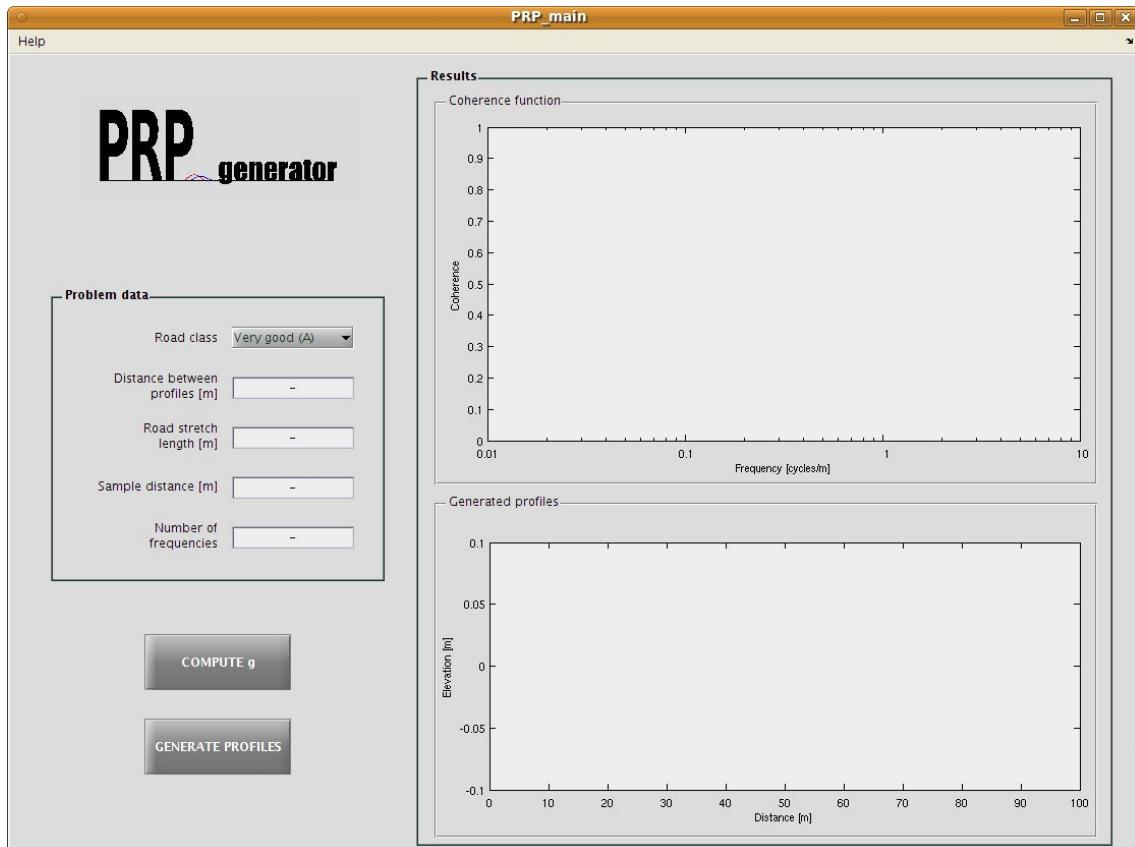


Figure 7: Initial appearance of the main and only window of PRPgenerator

The following parameters are set by the user in the *Problem data* section:

Road class according to ISO-8608. One option in the drop-down list has to be chosen; available road classes range from Very good (A) to Very poor (E).

Distance between profiles in meters. In general it will correspond with the vehicle width or, more precisely, with the axle track (distance between the centerline of two wheels on the same axle).

Road stretch length in meters. This is the length of the profiles to be generated.

Sample distance in meters. Since 0.1 m is the lower considered wavelength, the distance between samples in the generated profiles should not be higher than 0.02 m.

Number of frequencies (or equivalently wavelengths) to be used in the profile generation. This value corresponds to the N in expression (14). This value must be large enough if we want to get a good profile description, $N = 2000$ is a good choice.

There are two buttons on the lower left corner of the PRPgenerator window: **COMPUTE g** and **GENERATE PROFILES**:

- **COMPUTE g** calculates $G_x(n)$ and the coherence function $g(n)$. At the end of the computation, coherence function is depicted in the corresponding region of the *Results* section (see figure 8). Coherence function is stored in the working directory as an `ascii` file named `g_[Distance between profiles]_[Road class label].dat`. The Road class label is the value of $G(n_0)$ in table 1 multiplied by 10^6 , that is to say Class A=16, Class B=64, Class C=256, Class D=1024, Class E=4096. For example `g_2.2_16.dat` is the coherence function for a distance between profiles equal to 2.2 meters and Road class A. This file is organized into two columns: (1) spatial frequency n [m^{-1}] and (2) coherence function g . For the computation of $g(n)$ only Road class and Distance between profiles are needed so the other fields can remain unspecified at this point.
- **GENERATE PROFILES** does what it says. Road section length, sample distance and number of frequencies are needed now. When this button is clicked profiles are generated and depicted in the lower part of the *Results* section (figure 9). The values of g employed in the generation, that will depend on the Number of frequencies considered, are marked as black circles over the coherence function obtained before. Those values are saved in `G_GX_[Distance between profiles]_[Road class label].dat` along with the values of G and G_X . This file is organized into 4 columns: (1) Frequency n [m^{-1}], (2) G [m^3], (3) G_X [m^3] and (4) g . Profiles are saved in two files named `prof_[Distance between profiles]_[Road class label]_[Side label].dat` where Side label is LF for the left profile and RT for the right one. Profile files are organized in two columns: (1) Distance to the origin [m] and (2) Profile elevation [m].

If **GENERATE PROFILES** is clicked again a new pair of profiles is generated, depicted and saved. They will be stored in files with the same names as the previous profiles so these will be overwritten and therefore lost. If we want to create several profiles, as is usual, the procedure is: (1) Generate, (2) Protect the files by changing their names or moving them to another folder, (3) Generate another pair of profiles, (4) Repeat this actions as many times as necessary.

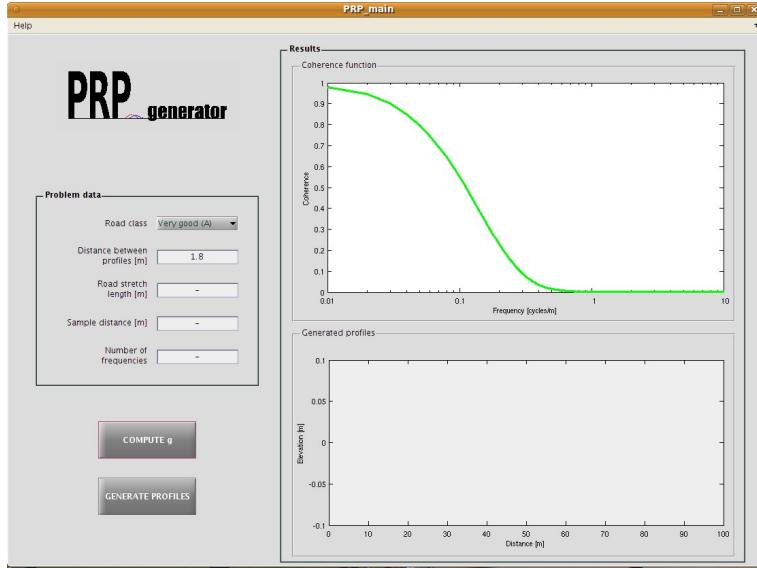


Figure 8: Once the coherence function is calculated by clicking **COMPUTE g** it is depicted in the upper region of the *Results* section using a logarithmic scale for the x-axis and a linear scale for the y-axis.

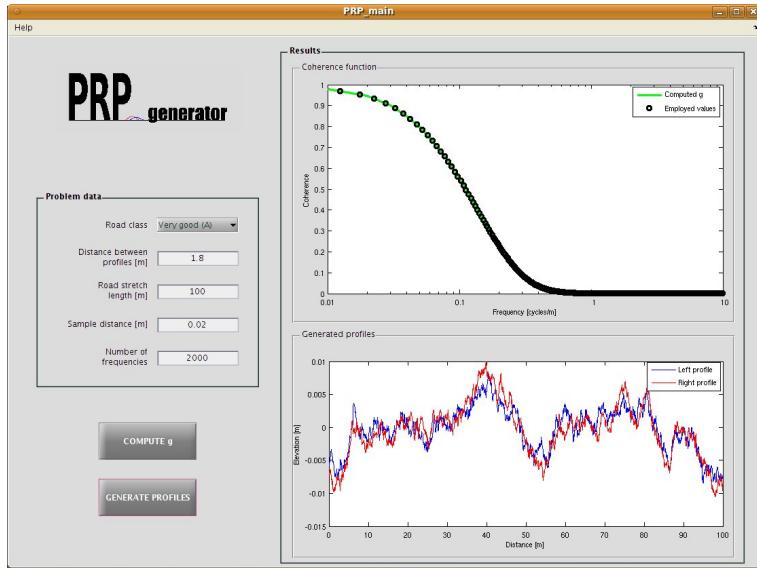


Figure 9: At the end of the profile generation the two parallel profiles are graphed in the lower part of the *Results* section, coherence function values for the spatial frequencies considered are marked in the coherence function graph.

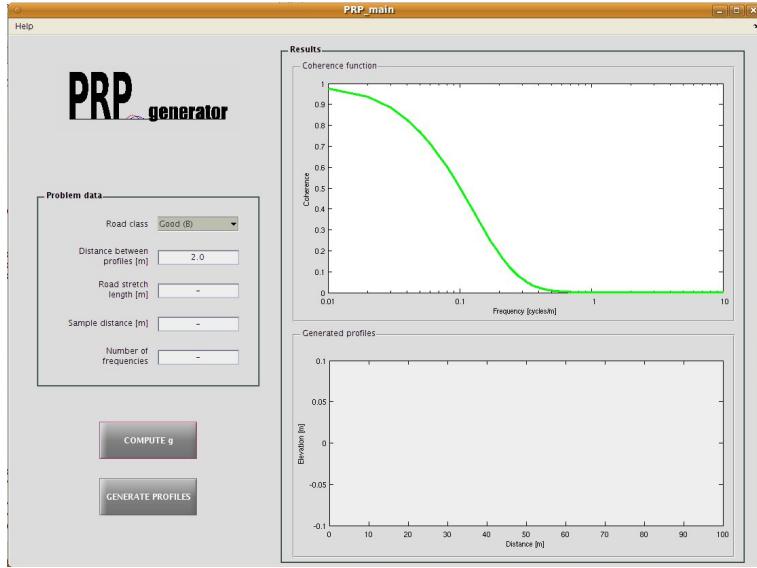


Figure 10: Tutorial: coherence function for Road class B and 2.0 meters between profiles is computed and depicted in PRPgenerator.

6 Tutorial

In this tutorial some parallel road profiles are generated with PRPgenerator. The aim of this work is to get the user acquainted with the software. The tasks are the following:

Generate three pairs of Class B profiles for a 2 meters wide vehicle:

1. *The first road stretch is 100 meters long, sample distance is set as 0.01 m and 2000 spatial frequencies are considered ($N = 2000$).*
2. *The second stretch is also 100 meters long but the sample distance is 0.02 m and only 1000 frequencies are employed ($N = 1000$).*
3. *The third stretch has the same features as the second one.*

In first place coherence function $g(n)$ for Class B profiles at 2.0 meters has to be computed. Select Road class Good (B) from the drop-down list and specify the distance between profiles that is the distance between left and right tyres of the same axle, 2.0 meters in this case. No other information is needed for the coherence function calculation. Now click **COMPUTE g** and wait (this part of the computation is lengthy). Once the progress bar reaches the end coherence function is graphed in the upper region of the *Results* section as shown in Figure 10. The file [g_2_64.dat](#) that contains the computed coherence function is then saved in the working directory.

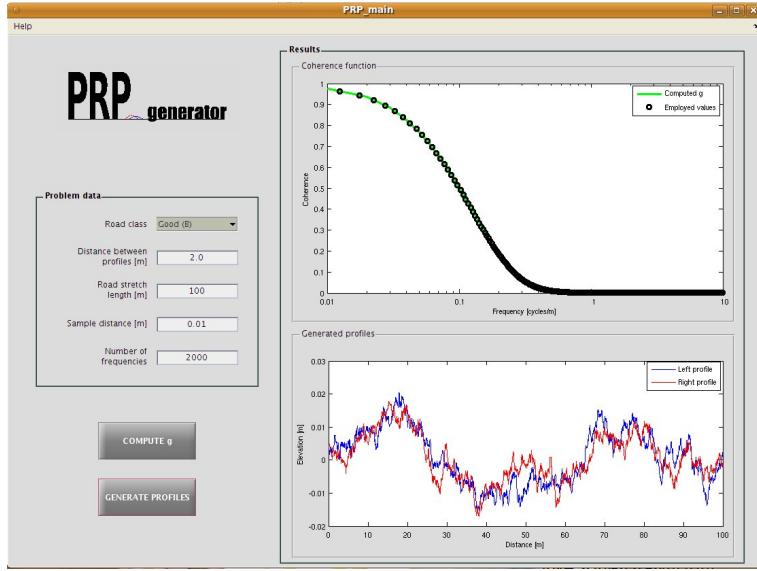


Figure 11: Tutorial: after clicking **GENERATE PROFILES** they are depicted in the bottom right-hand window, values of the coherence function employed in the generation are marked in the coherence function window. Notice that your upper graph must be the same as the one in this figure but not the lower one due to the randomness of the synthetic profiles.

Now you can generate your pairs of profiles. Let's start with the first profile. Set Road stretch length equal to 100 meters, Sample distance equal to 0.01 meters and a Number of frequencies of 2000. Click **GENERATE PROFILES**. Profile generation is much faster than coherence function computation so you do not have to be as patient as in the previous step. When the calculation is finished profiles are depicted in the lower part of the *Results* section and the employed values of $g(n)$ are marked over the coherence function graph (Figure 11). The software creates [G_GX_2_64.dat](#) in which the 2000 considered frequencies and the corresponding values of G , G_x and g are stored. Left profile is saved in [prof_2_64_LF.dat](#) and right profile in [prof_2_64_RT.dat](#).

Since we have been told to generate another pair of profiles we need to protect the just created files, otherwise they would be overwritten and the generated profiles lost. We may change their names or move them to another directory. Let's change their names as follows¹:

- [G_GX_2_64.dat](#) → [G_GX_2_64_2000.dat](#)
- [prof_2_64_LF.dat](#) → [prof_2_64_LF_2000.dat](#)
- [prof_2_64_RT.dat](#) → [prof_2_64_RT_2000.dat](#)

where **2000** indicates that we have used 2000 spatial frequencies.

In order to generate the second road section we need to change the values of sample distance (0.02 m) and number of frequencies (1000). Then click **GENERATE PROFILES**. The new profiles are depicted and the considered frequencies are shown

¹This is just an example, any notation can be adopted.

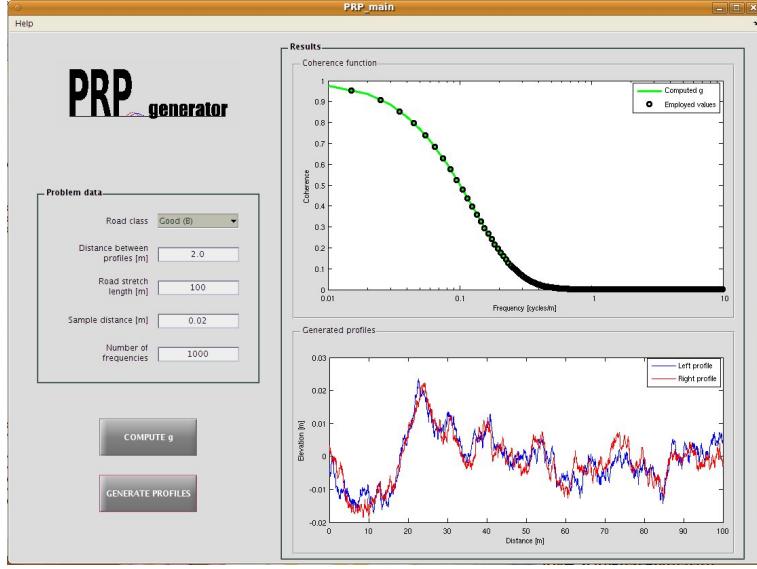


Figure 12: Tutorial: when the second pair of profiles is generated it is depicted and also are the values of $g(n)$ employed.

in the coherence function (Figure 12); notice that the considered frequencies are not the same as in the previous profile generation because we have changed the value of N . Values of n , G , G_x and g and the new profiles are stored in the corresponding files: [G_GX_2_64.dat](#), [prof_2_64_LF.dat](#) and [prof_2_64_RT.dat](#).

As we need a third set of parallel profiles we need to protect the just generated information. Let's change the file names again:

- [G_GX_2_64.dat](#) → [G_GX_2_64_1000_1.dat](#)
- [prof_2_64_LF.dat](#) → [prof_2_64_LF_1000_1.dat](#)
- [prof_2_64_RT.dat](#) → [prof_2_64_RT_1000_1.dat](#)

where [_1000_1](#) indicates that we have used 1000 spatial frequencies and this is the first set of profiles.

The last task is to obtain a new pair of profiles with the same features as the previous one. The only thing we have to do is to click **GENERATE PROFILES** again. A new pair of profiles is depicted and stored (Figure 13). If we want to keep a consistent file notation we shall rename the new files:

- [G_GX_2_64.dat](#) → [G_GX_2_64_1000_2.dat](#)
- [prof_2_64_LF.dat](#) → [prof_2_64_LF_1000_2.dat](#)
- [prof_2_64_RT.dat](#) → [prof_2_64_RT_1000_2.dat](#)

where [_1000_2](#) indicates that we have used 1000 spatial frequencies and this is the second set of profiles. It is important to notice that [G_GX_2_64_1000_2.dat](#) and [G_GX_2_64_1000_1.dat](#) will be exactly the same file since the Spatial frequencies considered have been the same in both calculations.

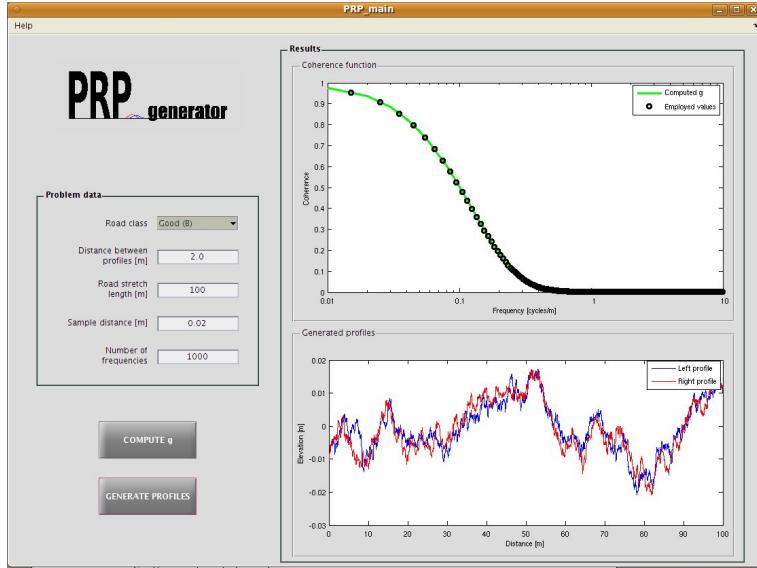


Figure 13: Tutorial: a new pair of profiles with the same properties as the previous one is generated and graphed if **GENERATE PROFILES** is clicked again.

6.1 Using a new working directory

If you want to work in another directory just copy PRPgenerator.exe (not PRPgenerator_pkg.exe) and the License folder into that directory and double click PRPgenerator.exe on that new working directory. Thus the software is run there and files created by PRPgenerator are saved to that folder.

References

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A Semi-analytical solution of the correlation functions

Second term integral in expression (10):

$$\int_{n_a}^{\infty} n^{-2} \cos(2\pi n \delta) dn \quad (15)$$

and second term integral in expression (11):

$$\int_{n_a}^{\infty} n^{-2} \cos(2\pi n \sqrt{\delta^2 + (2b)^2}) dn \quad (16)$$

are same kind integrals for each value of δ :

$$\int_{n_a}^{\infty} \frac{\cos(an)}{n^2} dn \quad (17)$$

where a is $a = 2\pi\delta$ in the first case and $a = 2\pi\sqrt{\delta^2 + (2b)^2}$ in the second one.

Therefore we need to integrate expression 17 and the solution will be applied to expressions (15) and (16).

Firstly we integrate by parts:

$$\begin{aligned} \int_{n_a}^{\infty} \frac{\cos(an)}{n^2} dn &= \left[-\frac{\cos(an)}{n} \right]_{n_a}^{\infty} - a \int_{n_a}^{\infty} \frac{\sin(an)}{n} dn \\ &= \frac{\cos(an_a)}{n_a} - a \int_{n_a}^{\infty} \frac{\sin(an)}{n} dn \end{aligned} \quad (18)$$

So now the integral to be solved is:

$$\int_{n_a}^{\infty} \frac{\sin(an)}{n} dn \quad (19)$$

that can be divided into two parts as follows:

$$\int_{n_a}^{\infty} \frac{\sin(an)}{n} dn = \int_0^{\infty} \frac{\sin(an)}{n} dn - \int_0^{n_a} \frac{\sin(an)}{n} dn \quad (20)$$

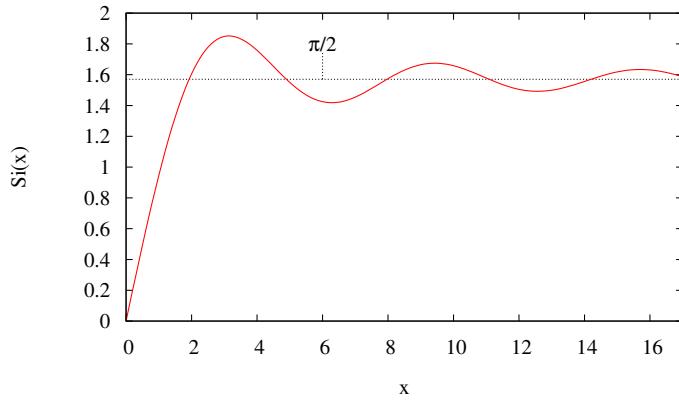


Figure 14: Dirichlet Integral.

Let's stop here to study the integral of function $\frac{\sin(n)}{n}$. Indefinite integral of this function does not exist in closed-form. Nevertheless, the definite integral with upper and lower limits equal to 0 and x respectively defines a new function of x known as *Sine Integral*:

$$Si(x) = \int_0^x \frac{\sin(n)}{n} dn \quad (21)$$

We can approximate this function by using McLaurin series expansion of $\sin(n)$:

$$\sin(n) \simeq n - \frac{n^3}{3!} + \frac{n^5}{5!} - \dots \quad (22)$$

The integrand turns into:

$$\frac{\sin(n)}{n} \simeq 1 - \frac{n^2}{3!} + \frac{n^4}{5!} - \dots \quad (23)$$

And the integral becomes:

$$Si(x) = \int_0^x \frac{\sin(n)}{n} dn \simeq \int_0^x \left(1 - \frac{n^2}{3!} + \frac{n^4}{5!} - \dots \right) dn = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \dots \quad (24)$$

a series that converges for all x .

If we plot this approximation (figure 14) we can see that the sine integral tends to $\frac{\pi}{2}$ when x tends to ∞ . Hence:

$$\lim_{x \rightarrow \infty} Si(x) = \int_0^\infty \frac{\sin(n)}{n} dn = \frac{\pi}{2} \quad (25)$$

This expression is known as *Dirichlet integral*, after the German mathematician Johann Peter Gustav Lejeune Dirichlet²

If we substitute $\sin(n)$ with $\sin(an)$ where a is constant it is easy to demonstrate the following:

$$\int_0^\infty \frac{\sin(an)}{n} dn = \begin{cases} \frac{\pi}{2} & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -\frac{\pi}{2} & \text{if } a < 0 \end{cases} \quad (26)$$

²Actually, there are several integrals with this name; this is one of them.

It is trivial to prove that a will always be positive or null.

Let's go back to the expression 20:

$$\int_{n_a}^{\infty} \frac{\sin(an)}{n} dn = \int_0^{\infty} \frac{\sin(an)}{n} dn - \int_0^{n_a} \frac{\sin(an)}{n} dn$$

The first term in the right-hand is solved by means of equation 26. The second term can be approximated by using McLaurin series expansion:

$$\int_0^{n_a} \frac{\sin(an)}{n} dn = an_a - \frac{(an_a)^3}{3 \cdot 3!} + \frac{(an_a)^5}{5 \cdot 5!} - \dots = \text{Si}(an_a) \quad (27)$$

Substituting equations 26 and 27 in expression 20:

$$\int_{n_a}^{\infty} \frac{\sin(an)}{n} dn = \begin{cases} \frac{\pi}{2} - \text{Si}(an_a) & \text{si } a > 0 \\ 0 - \text{Si}(an_a) & \text{si } a = 0 \end{cases} \quad (28)$$

where $\text{Si}(an_a)$ is computed by means of McLaurin series expansion.

With expression (28) we can solve the initial integral (equation (18)):

$$\int_{n_a}^{\infty} \frac{\cos(an)}{n^2} dn = \frac{\cos(an_a)}{n_a} - a \left[\frac{\pi}{2} - \text{Si}(an_a) \right] \quad (29)$$

It is important to notice that the piecewise expression in equation (28) vanished in equation (29), the reason is that the second term in the right-hand side will be always zero if $a = 0$ so it is not necessary to make a distinction and the expression can be simplified.

Therefore, integral in expression (15) where $a = 2\pi\delta$ is solved as follows:

$$\int_{n_a}^{\infty} \frac{\cos(2\pi n \delta)}{n^2} dn = \frac{\cos(2\pi n_a \delta)}{n_a} - 2\pi\delta \left[\frac{\pi}{2} - \text{Si}(2\pi n_a \delta) \right] \quad (30)$$

And integral in expression (16) in which $a = 2\pi\sqrt{\delta^2 + (2b)^2}$:

$$\begin{aligned} \int_{n_a}^{\infty} \frac{\cos(2\pi n \sqrt{\delta^2 + (2b)^2})}{n^2} dn &= \\ &= \frac{\cos(2\pi n_a \sqrt{\delta^2 + (2b)^2})}{n_a} - 2\pi\sqrt{\delta^2 + (2b)^2} \left[\frac{\pi}{2} - \text{Si}(2\pi n_a \sqrt{\delta^2 + (2b)^2}) \right] \end{aligned} \quad (31)$$